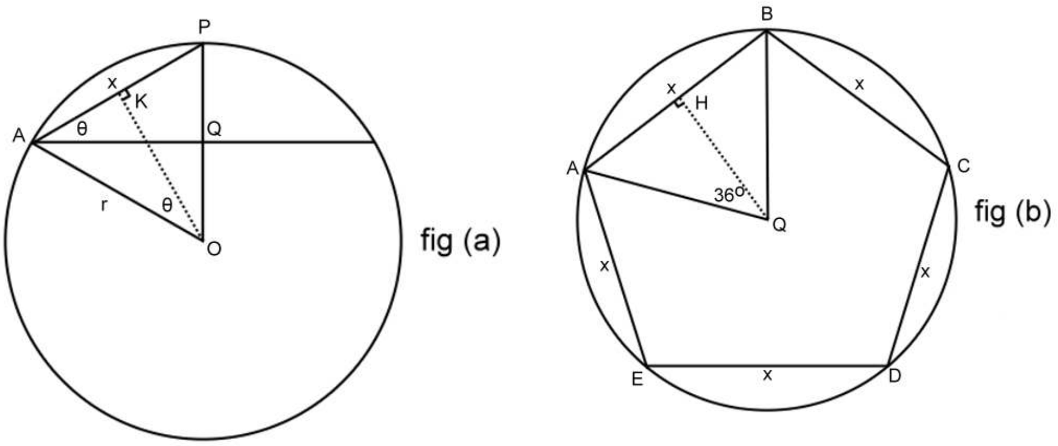
The Geometry of the 'Singapore Ball'

While chatting to a friend of mine, John Haigh, who is a mathematician, the subject of the spacing of holes on a Singapore ball arose (like it does).

I'd shown John a Singapore ball and said that to find the compass setting for marking out the first 12 holes I knew I'd to multiply the diameter of the ball by 0.526. I realised this multiplier (0.526) must arise from the geometry of a sphere but didn't know how exactly. I asked John if he could explain it to me, so that when people asked (as they do from time to time) I'd have an answer for them. What follows is John's explanation:

A Singapore ball is a solid sphere into which 32 holes have been drilled. The first five holes mark the vertices of a regular icosahedron inscribed in the sphere. An icosahedron has 20 triangular faces, 12 vertices and 30 edges (Greek '*eikosi* ' = 20).

If the first hole (labelled P in fig.(a)) is taken to define an axis OP through the centre of the sphere O, then the next five holes form a regular pentagon $ABCDE$ in a plane perpendicular to the axis OP as shown in fig.(b).



The axis OP meets the plane in a point Q, which is the centre of the pentagon and its escribing circle. The radius *r* of the ball is known, but we must determine x', the length of (any) edge of the icosahedron.

Now $\angle AQH = \dfrac{360^o}{10} = 36^o$ where H is the mid-point of edge AB

$$\therefore \frac{AH}{AQ} = \sin 36^o$$

but $AH = \frac{1}{2}x$ (half the length of edge AB)

$$\therefore AQ = \frac{x}{2\sin 36^o} = \alpha\, x \qquad \text{where } \alpha = \frac{1}{2\sin 36^o}$$

In fig. (a) we know that:

$AP = x$	$AQ = \alpha\, x$	$AO = r$
(edge length)		(radius of sphere)

One way to determine x from this information is to apply Pythagoras' theorem to $\Delta\, AQP$ to determine PQ, and again to $\Delta\, AQO$ to determine QO and then note that $PQ + QO = PO = r$.

However, possibly the neatest way is to quote a geometric theorem which states that the angle subtended by a chord of a circle at the centre is twice the angle subtended at any point on the opposite arc.

From which it follows that:

$\angle\, PAQ = \angle\, AOK$ where K is the mid-point of AP

Both of these angles are labelled θ in fig. (b).

From $\Delta\, AQP$, $\cos\theta = \frac{AQ}{AP} = \alpha$

From $\Delta\, AKO$, $\sin\theta = \frac{AK}{AO} = \frac{\frac{1}{2}x}{r} = \frac{x}{2r}$

But, $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \quad \alpha^2 + \left(\frac{x}{2r}\right)^2 = 1$$

$$\therefore \quad x^2 = 4r^2(1 - \alpha^2)$$

$$\therefore \quad x = 2r\sqrt{1 - \frac{1}{4\sin^2 36^o}}$$

$$\therefore \quad x = 2r \times 0.525731112$$

Hence the circle containing the pentagon $ABCDE$ can be determined with a compass set to 0.526 (to three decimal places) of the ball's diameter $(2r)$.

The remaining 20 points are obtained by drilling an additional hole above the centre *(i.e. on the surface of the ball)* of each of the 20 faces of the icosahedron. These correspond to the vertices of the regular dodecahedron which is the dual of the icosahedron (12 faces, 20 vertices, 30 edges).

Well, there it is. Multiply the diameter by 0.526, set your compasses to the resulting figure and mark out the first 12 holes. Mark the other 20 holes by finding the centres of the 20 triangular faces by a 'trial and error' method using compasses. John also went on to provide an interesting addendum:

If you divide each of the 30 edges of the icosahedron into three equal parts and use the points of trisection to surround each of the 12 icosahedral vertices with a pentagon, you will simultaneously surround each of the 20 dodecahedral vertices with a hexagon. In fact you will obtain the pattern which is often displayed on the surface of a modern soccer ball.

If you devise some way to colour the areas surrounding each of the drill points on the Singapore ball, you might have an unusual but appropriate trophy for soccer competitions.

The semi-regular solid with 32 pentagonal or hexagonal faces, 60 vertices at the trisection points of the icosahedral edges, and 90 edges (of ⅓ length), defines the structure of 'Buckminster Fullerene', or the 'bucky ball' (C_{60}), the third stable allotrope of carbon.